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Turbulent pair dispersion of inertial particles

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The relative dispersion of pairs of inertial point particles in incompressible, homogeneous and isotropic three-dimensional turbulence is studied by means of direct numerical simulations at two values of the Taylor-scale Reynolds number $Re_\lambda \sim 200$ and $Re_\lambda \sim 400$, corresponding to resolutions of 512^3 and 2048^3 grid points, respectively. The evolution of both heavy and light particle pairs is analysed by varying the particle Stokes number and the fluid-to-particle density ratio. For particles much heavier than the fluid, the range of available Stokes numbers is $St \in [0.1 : 70]$, while for light particles the Stokes numbers span the range $St \in [0.1 : 3]$ and the density ratio is varied up to the limit of vanishing particle density. For heavy particles, it is found that turbulent dispersion is schematically governed by two temporal regimes. The first is dominated by the presence, at large Stokes numbers, of small-scale caustics in the particle velocity statistics, and it lasts until heavy particle velocities have relaxed towards the underlying flow velocities. At such large scales, a second regime starts where heavy particles separate as tracers' particles would do. As a consequence, at increasing inertia, a larger transient stage is observed, and the Richardson diffusion of simple tracers is recovered only at large times and large scales. These features also arise from a statistical closure of the equation of motion for heavy particle separation that is proposed and is supported by the numerical results. In the case of light particles with high density ratio, strong small-scale clustering leads to a considerable fraction of pairs that do not separate at all, although the mean separation increases with time. This effect strongly alters the shape of the probability density function of light particle separations.

1. Introduction

Suspensions of dust, droplets, bubbles and other finite-size particles advected by incompressible turbulent flows are commonly encountered in many natural phenomena (see, e.g. Csanady 1980; Eaton & Fessler 1994; Falkovich, Fouxon & Stepanov 2002; Post & Abraham 2002; Shaw 2003; Toschi & Bodenschatz 2009).

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Understanding their statistical properties is, thus, of primary importance. From a theoretical point of view, the problem is more complicated than that in the case of fluid tracers, i.e. point-like particles with the same density as the carrier fluid. Indeed, when the suspended particles have a finite size and a density ratio different from that of the fluid, they have inertia and do not follow exactly the flow. As a consequence, correlations between particle positions and structures of the underlying flow appear. It is, for instance, well known that heavy particles are expelled from vortical structures, while light particles tend to concentrate in their cores. This results in the formation of strong inhomogeneities in the particle spatial distribution, an effect often referred to as *preferential concentration* (see Douady, Couder & Brachet 1991; Squires & Eaton 1991; Eaton & Fessler 1994). This phenomenon has gathered much attention, as it is revealed by the amount of recently published theoretical work (Balkovsky, Falkovich & Fouxon 2001; Zaichik, Simonin & Alipchenkov 2003; Falkovich & Pumir 2004) and numerical studies (Reade & Collins 2000; Collins & Keswani 2004; Chun *et al.* 2005; Bec *et al.* 2007*a*; Goto & Vassilicos 2008). Progresses in the statistical characterization of small particle aggregates have been achieved by studying particles evolving in stochastic flows by Sigurgeirsson & Stuart (2002), Mehlig & Wilkinson (2004), Bec *et al.* (2005) and Olla (2002), and in two-dimensional turbulent flows by Boffetta, De Lillo & Gamba (2004). Also, single trajectory statistics have been addressed both numerically and experimentally for small, heavy particles and light particles (see, e.g. Bec *et al.* 2006; Cencini *et al.* 2006; Gylfason *et al.* 2006; Ayyalasomayajula, Warhaft & Collins 2008; Gerashchenko *et al.* 2008; Zaichik & Alipchenkov 2008; Volk *et al.* 2008, 2009), and for large particles, where inertial effects combine with finite size ones (see e.g. Qureshi *et al.* 2007, 2008; Xu & Bodenschatz 2008). The reader is referred to Toschi & Bodenschatz (2009) for a review.

In this paper we are concerned with small (point) particle pair dispersion, that is with the statistics, as a function of time, of the separation distance $\boldsymbol{R}(t) = \boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)$ between two inertial particles, labelled by the subscripts 1 and 2. While there is a vast literature on the fluid tracer dispersion in turbulent flows (recent reviews are Sawford 2001 and Salazar & Collins 2009), much less is known about inertial particles. For the case of heavy suspensions, the problem has been previously investigated analytically in random flows (Derevyanko *et al.* 2007; Bec *et al.* 2008; Derevich 2008; Fouxon & Horvai 2008), by means of kinematic simulations (El Maihy & Nicolleau 2005), and also in direct numerical simulations of thermally stratified turbulence (van Aartrijk & Clercx 2009).

In homogeneous turbulence, it is sufficient to consider the statistics of the instantaneous separation of the positions of the two particles. These are organized in different families according to the values of their Stokes number St and *density mismatch* with the fluid, β. The Stokes number is defined as $St = \tau_s/\tau_\eta$, where the particle response time is $\tau_s = a^2/(3\nu\beta)$, a being the particle radius much smaller than the Kolmogorv scale η, and where τ_η is the flow Kolmogorov time scale. The adimensional constant $\beta = 3\rho_f/(\rho_f + 2\rho_p)$ accounts for the contrast between particle density ρ_p and fluid density ρ_f.

For our purposes, the motion of particle pairs, with given (St, β) values and with initial separations inside a given spherical shell, $R = |\boldsymbol{X}_1(t_0) - \boldsymbol{X}_2(t_0)| \in [R_0, R_0 + dR_0]$, is followed until particle separation reaches the large scale of the flow. With respect to the case of simple tracers, the time evolution of the inertial particle pair separation $R(t)$ becomes a function not only of the initial distance R_0 and the Reynolds number of the flow but also of the inertia parameters (St, β).

A key question that naturally arises is how to choose the initial spatial and velocity distributions of inertial pairs. It is known that heavy (respectively light) particles tend to concentrate preferentially in hyperbolic (respectively elliptic) regions of the advecting flow; indeed, spatial correlation effects may extend up to the inertial range of scales for not too large Stokes numbers, as shown by Bec *et al.* (2007*a*), Chen, Goto & Vassilicos (2006) and Calzavarini *et al.* (2008*a*). Moreover, when inertia is high enough, the particle pair velocity difference, $\delta_R V = |\boldsymbol{V}_1(\boldsymbol{X}_1(t), t) - \boldsymbol{V}_2(\boldsymbol{X}_2(t), t)|$, may not go smoothly to zero when the particle separations decrease, a phenomenon connected to the formation of *caustics* (see Wilkinson & Mehlig 2005; Falkovich & Pumir 2007). This implies for example that if inertia is such that particle response time is larger than any turbulent flow time scale, nearby particles will move with uncorrelated velocities (see e.g. Abrahamson 1975; Simonin, Février & Laviéville 2002; Bec, Cencini & Hillerbrand 2007*b*).

In our numerical simulations, particles of different inertia are injected into the flow and evolve until they reach a stationary statistics for both spatial and velocity distributions. Only after this transient time, pairs of particles with fixed initial separation are selected and then followed in the spatial domain to study relative dispersion.

In view of the above considerations, the main issue is to understand the role played by the spatial inhomogeneities of the inertial particle concentration field and by the presence of caustics on the pair separations, at changing the degree of inertia. We remark that these two effects can be treated as independent only in the limit of very small and very large inertia. In the former case, particles tend to behave like tracers and move with the underlying fluid velocity: preferential concentration may affect only their separation. In the opposite limit, particles distribute almost homogeneously in the flow; however, due to their ballistic motion, they can reach nearby positions with very different velocities (Falkovich *et al.* 2002). In any other case of intermediate inertia, both of these effects are present and may play a role in the statistics of inertial pair separation.

It is worth anticipating the two main results of the present study, which are as follows.

(i) The separation between heavy particles can be described in terms of two time regimes: the first regime is dominated by inertia effects, and considerable deviations from the tracers arise in the inertial relative dispersion; in the second one, the tracers' behaviour is recovered since inertia is weak and appears only in subdominant corrections that vanish as $1/t$. The crossover between these two regimes defines new characteristic spatial and temporal scales, connected to both the *range of scale of caustics*, and the Stokes number, which influence the particle separation for not too long time-lags and not too large scales.

(ii) The strong clustering properties that are typical of light particles may lead to the fact that many pairs do not separate at all: their statistical weight is clear in the separation probability density function (PDF), which develops a well-defined power-law left tail.

It would also be interesting to investigate the dependence upon the Reynolds number of the inertial particle pair separation. Small-scale clustering seems to be poorly dependent on the degree of turbulence of the carrier flow (Collins & Keswani 2004; Bec *et al.* 2007*a*), while much less is known about the Reynolds number dependence of the caustics statistics. Our numerical data do not allow to explore this question in detail, so we will restrict ourselves to show data associated to the two Reynolds numbers in all cases when differences are not significative.

	N	Re_λ	η	δx	ε	ν	τ_η	t_{dump}	δt	T_L
Run I	512	185	0.01	0.012	0.9	0.002	0.047	0.004	0.0004	2.2
Run II	2048	400	0.0026	0.003	0.88	0.00035	0.02	0.00115	0.000115	2.2

TABLE 1. Eulerian parameters for the two runs analysed here: Run I and Run II in the text. N is the number of grid points in each spatial direction; Re_λ is the Taylor-scale Reynolds number; η is the Kolmogorov dissipative scale; $\delta x = \mathscr{L}/N$ is the grid spacing, with $\mathscr{L} = 2\pi$ denoting the physical size of the numerical domain; $\tau_\eta = (\nu/\varepsilon)^{1/2}$ is the Kolmogorov dissipative time scale; ε is the average rate of energy injection; ν is the kinematic viscosity; τ_{dump} is the time interval between two successive dumps along particle trajectories; δt is the time step; $T_L = L/U_0$ is the eddy turnover time at the integral scale $L = \pi$ and U_0 is the typical large-scale velocity.

In the case of fluid tracers, the standard observables are the time evolution of the mean-square separation and the separation PDF, for which well-established predictions exist since the pioneering work of Richardson (1926). We contrast these observables obtained for tracers with the results for heavy and light inertial particles.

The paper is organized as follows. In § 2, we briefly recall the basic equations of motion and describe the numerical simulations. In § 3, we analyse the stationary distribution of heavy particle velocity differences, conditioned on the particle initial separation, highlighting both the presence of small-scale caustics and the effects of particle inertia at scales corresponding to the inertial range of fluid turbulence. In § 4, we study the behaviour of the mean separation distance of heavy pairs at changing the Stokes number St; we also analyse the influence of the caustics in the initial statistics on the subsequent pair separation evolution. A *mean-field* model, which is able to capture the main numerical findings, is proposed in the same section. The time evolution of the separation PDFs is discussed in § 5 and we present the data for light particles in § 6. In § 7 we summarize the main findings.

2. Equation of motion and numerical details

We present results from direct numerical simulations of three-dimensional turbulent flows seeded with inertial particles. The flow phase is described by the Navier–Stokes equations for the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{2.1}$$

The statistically homogeneous and isotropic external forcing $\boldsymbol{f}$ injects energy in the first low wavenumber shells, by keeping constant their spectral content (see Chen *et al.* 1993). The kinematic viscosity ν is chosen such that the Kolmogorov length scale $\eta \approx \delta x$, where δx is the grid spacing; this choice ensures a good resolution of the small-scale velocity dynamics. The numerical domain is cubic and 2π-periodic in the three directions of space. We use a fully dealiased pseudospectral algorithm with second-order Adam–Bashforth time-stepping (for details see Bec *et al.* 2006; Cencini *et al.* 2006). We performed two series of DNSs: Run I with numerical resolution of 512^3 grid points and the Reynolds number at the Taylor scale $Re_\lambda \approx 200$; Run II with 2048^3 resolution and $Re_\lambda \approx 400$. Details of the runs can be found in table 1.

The particle phase is constituted by millions of heavy and light particles — the latter only for Run I — with different intrinsic characteristics. Particles are assumed to be with size much smaller than the Kolmogorov scale of the flow — numerically, they are treated as point particles—and with a negligible Reynolds number relative to the particle size. In this limit, the equations ruling their dynamics take the particularly simple form:

$$\dot{\boldsymbol{X}} = \boldsymbol{V}, \quad \dot{\boldsymbol{V}} = -\frac{1}{\tau_s}\left[\boldsymbol{V} - \boldsymbol{u}(\boldsymbol{X}, t)\right] + \beta\, D_t\boldsymbol{u}(\boldsymbol{X}, t), \tag{2.2}$$

where the dots denote time derivatives. The particle position and velocity are $\boldsymbol{X}(t)$ and $\boldsymbol{V}(t)$, respectively; $\boldsymbol{u}(\boldsymbol{X}(t), t)$ is the Eulerian fluid velocity evaluated at the particle position, and $D_t\boldsymbol{u}$ is the so-called added mass term, which measures the fluid acceleration along particle trajectory.

The adimensional constant β accounts for the added mass effect resulting from the density contrast of the particles with the fluid. The particle response time, appearing in the Stokes drag, is τ_s. The flow Kolmogorov time scale, appearing in the definition of the Stokes number $St = \tau_s/\tau_\eta$, is $\tau_\eta = (\nu/\varepsilon)^{1/2}$, where ν is the flow kinematic viscosity and ε is the average rate of energy injection.

Equation (2.2) has been derived under the assumption of very dilute suspensions, where particle–particle interactions (collisions) and the feedback of the particles onto the flow can be neglected (see, e.g. Maxey & Riley 1983 for a discussion of the complete equation of motion of a small spherical particle in a non-uniform flow).

For Run I, we show results for the following set of (St, β) families: (i) very heavy particles $[\beta = 0]$: $St = 0.0, 0.6, 1.0, 3.3$; (ii) light particles $[\beta = 2, 3]$: $St = 0.3, 1.2, 4.1$. For each family, the typical number of particle pairs that are followed is around 5×10^4. For Run II, we show results only for heavy particles but with a larger range of variation in the Stokes number: $St = 0.0, 0.6, 1.0, 3.0, 10, 30, 70$. The typical number of particle pairs for each family is now $\sim 10^4$.

Once injected particles have relaxed to their steady-state statistics, pairs have been selected with the following initial separations: $R_0 \leqslant \eta$ and $R_0 \in [4:6]\eta$ for both Run I and Run II, and $R_0 \in [9:11]\eta$ for Run II only.

Besides the time evolution of particle pairs, we collected instantaneous snapshots of the two phases (fluid and dispersed), with a much higher particle statistics: around 10^6 per family for Run I and 10^8 per family for Run II. These are used to measure the stationary — i.e. not along the trajectories — distribution of particle velocity increments discussed in the next section.

3. Stationary distributions: velocity increments conditioned on particles separation

Turbulent pair dispersion for tracers is classically based on the application of similarity theory for Eulerian velocity statistics; depending on the value of space and time scales, velocity increment statistics differently affect the way tracers separate. This results in different regimes for relative dispersion (see e.g. Sawford 2001).

In the case of inertial particles, a similar reasoning holds, so to analyse the way inertial pairs separate in time, the stationary statistics of particle velocity differences have to be investigated first. A stationary distribution for the typical velocity differences between two inertial particles is obtained by imposing periodic boundary conditions inside the physical volume and then measuring velocities on such a thermalized configuration.

We are interested in the scaling behaviour of velocity increments at varying the degree of inertia and the distance between the particles (in the dissipative or inertial range of the homogeneous and isotropic turbulent fluid flow). To fix the notation, we denote by U_0 the typical large-scale velocity of the fluid tracers and by L the integral

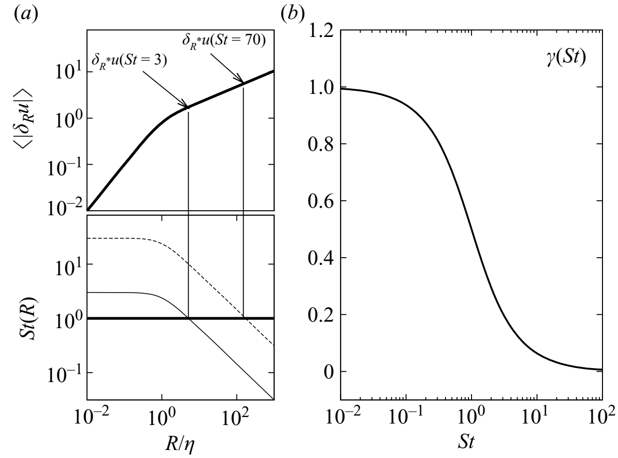

FIGURE 1. (a, bottom) Behaviour of the scale-dependent Stokes number, $St(R)$, as a function of the scale R normalized with the Kolmogorov scale η, for two Stokes numbers $St=3, 70$ (bottom and top, respectively). The horizontal thick line is for $St(R)=1$. (a, top) The scaling behaviour for the fluid tracer velocity increments vs. the scale as given by (3.2). Note that the scales R^* where $St(R^*)=1$ fall in the inertial range of the Eulerian fluid velocity. (b) The function $\gamma(St)$, fitted from the data, defining the small-scales power-law behaviour of caustics statistics at changing inertia. Note that for small values of the Stokes number St, $\gamma(St)\to 1$, i.e. particle velocity is differentiable; at high inertia, $\gamma(St)\to 0$, indicating the existence of discontinuities in the particle velocity increment statistics.

scale of the flow. Moreover, we define

$$\delta_R V_{St} = |\boldsymbol{V}_1(\boldsymbol{X}_1(t)) - \boldsymbol{V}_2(\boldsymbol{X}_2(t))| \tag{3.1}$$

as the velocity difference at scale R, conditioned on the presence of a pair of particles with Stokes number St, separated with a distance $R=|\boldsymbol{X}_1(t)-\boldsymbol{X}_2(t)|$. Since we are interested in the case of heavy particles only, the Stokes number is sufficient to identify a given particle family. Moreover, since we are considering the stationary statistics, we can drop the time dependence. For convenience, we introduce a specific notation for the tracer stationary velocity statistics: $\delta_R u = \delta_R V_{(St=0)}$, which is exactly equal to the Eulerian velocity increment at scale R.

Recently, Bec *et al.* (2008) have shown that to describe inertial pair dispersion in synthetic flows, it is useful to introduce the local or *scale-dependent* Stokes number. This is defined as the ratio between the particle response time and the typical eddy turnover time $\tau_R = R/\delta_R u$ of the underlying fluid at a given scale R: $St(R)=\tau_s/\tau_R \sim \tau_s\,\delta_R u/R$. For real turbulent flows where different scaling ranges are present, we can equivalently define a scale-dependent Stokes number $St(R)$ that recovers the usual definition of the Stokes number, $St(R)\simeq St=\tau_s/\tau_\eta$ when $R\ll\eta$, and behaves as $St(R)\sim \tau_s\varepsilon^{1/3}R^{-2/3}$ when $\eta \ll R \ll L$. The typical behaviour of $St(R)$ is sketched in figure 1 for two different values of the Stokes number $St=3, 70$ and using a Batchelor-like parametrization of the fluid velocity (see Meneveau 1996):

$$\delta_R u = U_0 \frac{R}{(\eta^2+R^2)^{1/3}}. \tag{3.2}$$

For Stokes numbers, St, order unity or larger, there always exists a typical scale where the local Stokes number, $St(R)$, becomes order unity (El Maihy & Nicolleau

2005):

$$R^*(St) = \eta\, St^{3/2}. \tag{3.3}$$

Such a scale, which is well in the inertial range if the Stokes number St is sufficiently large, can be considered a rough estimate of the upper bound for the region of scales where inertia plays an important role in the particle dynamics. We expect that two main features might be important in characterizing the inertial particle stationary velocity statistics $\delta_R V_{st}$, with respect to that of tracers $\delta_R u$. The first concerns the small-scale behaviour of the particle velocity statistics. At small scales $R \ll \eta$ and for large-enough Stokes numbers, the presence of caustics makes the particle velocity increments not differentiable. This feature can be accounted for by saying that

$$\delta_R V_{St} \sim V^{\eta}_{St} \left(\frac{R}{\eta}\right)^{\gamma(St)}, \quad R \ll \eta, \tag{3.4}$$

where V^{η}_{St} is a constant prefactor and the function $\gamma(St)$ (introduced by Bec *et al.* 2005) gives the typical scaling of caustic-like velocity increments. Indeed, we do expect that at changing the inertia of the particles, the statistical weight of caustics might monotonically vary as follows. At small St, $\lim_{St\to 0} \gamma(St) = 1$, i.e. the value for smooth, differentiable Eulerian statistics of tracers. At large values $St \to \infty$, it should approach the discontinuous limit $\gamma(St) \to 0$, valid for particles that do not feel underlying fluid fluctuations at all. Figure 1(*b*) shows the typical shape of the function $\gamma(St)$ obtained by fitting the data at the two available Reynolds numbers (Bec *et al.* 2009): the functional form fitting the data is $\gamma(St) = [1 - 2/\pi \arctan(g_1\, St)]$, where g_1 is an adimensional constant order unity.

The second important feature concerns the particle velocity statistics at scales larger than the scale $R^*(St)$ previously defined, but smaller than the integral scale of the fluid flow. For any fixed Stokes number and a large-enough Reynolds number, we expect that inertia becomes weaker and weaker, by going to larger and larger scales $R \gg R^*(St)$. In such a case, particle velocity increments are expected to approach the underlying fluid velocity increments:

$$\delta_R V_{St} \to V^0_{St}\, \delta_R u \sim V^0_{St}\, U_0 \left(\frac{R}{L}\right)^{1/3}, \quad R^*(St) \ll R \ll L, \tag{3.5}$$

where for simplicity we have neglected possible intermittent correction to the Kolmogorov's 1941 (K41) scaling of the fluid velocity (see Frisch 1995 for details). Clearly, the Reynolds number has to be sufficiently large to provide a well-developed scaling region $R^*(St) \ll R \ll L$, before approaching the large scale L. We emphasize that in (3.5), an adimensional normalization factor V^0_{St} has been introduced, which takes into account possible filtering effects induced by inertia at large scales. The normalization is such that $V^0_{(St=0)} = 1$, while that for any Stokes larger than zero $V^0_{(St)} \leqslant 1$.

In figure 2, we test the validity of the previous picture by analysing the typical velocity fluctuation, $\langle|\delta_R V_{St}|\rangle$, at changing Stokes number and for data of Run II at Reynolds number $Re_\lambda \sim 400$. At small scales, the presence of caustics in the velocity statistics can be detected, with a non-smooth scaling behaviour below the Kolmogorov scale η. At scales within the inertial range and when the Stokes number is sufficiently large, the presence of caustics also affects particle velocity statistics up to a characteristic scale that becomes larger and larger by increasing particle inertia. Beyond this scale, particle velocity increments tend to approach the scaling behaviour of the fluid tracers, but their amplitude is depleted by a factor $1/V^0_{St}$, which increases

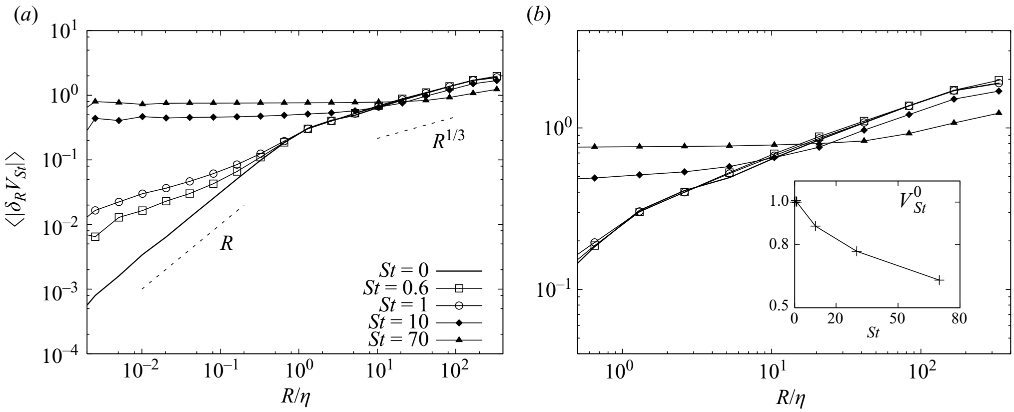

FIGURE 2. (*a*) Particle velocity structure function of order $p=1$ vs. the scale R/η, for various Stokes numbers, $St=0, 0.6, 1, 10, 70$, and for Reynolds number $Re_\lambda \sim 400$, Run II. The statistics for fluid tracers ($St=0$) correspond to the solid line. Statistical errors are of the order of twice the size of symbols for scales smaller than η and become comparable with the size of symbols in the inertial range of fluid velocity statistics. The differentiable scaling behaviour $\propto R$ in the dissipative range and the Kolmogorov 1941 behaviour $\propto R^{1/3}$ in the inertial range of scales are also shown. (*b*) Zoom-in of the inertial range, symbols are the same as in (*a*). (Inset) Behaviour of the amplitude prefactor, V^0_{St}, as a function of the Stokes number St, as measured from the velocity increments at the integral scale L, Run II.

with the Stokes number, as shown in the inset of figure 2(*b*). If we neglect the role of intermittency, a similar behaviour is expected for higher-order fluctuations.

It is interesting to consider the scaling behaviour of particle velocity in terms of the underlying velocity statistics, not only at very small or very large separations, but also for any value of the scale R. This is not straightforward, since we have to take into account not only the fluid Eulerian statistics at the dissipative and inertial range of scales, but also the modifications due to the inertia. This is responsible, as we have seen, for the appearance of a new relevant scale and for filtering effects in the velocity amplitude.

To fully characterize particle velocity increments, we note that the Stokes scale, $R^*(St)$ defines a typical *Stokes velocity*: this is the fluid velocity increment at the Stokes scale, $\delta u^*(St) \sim \delta_{R^*} u$ (see figure 1*a*). Previous reasonings can be summarized in the following interpolation formula for the heavy particle velocity increment:

$$\delta_R V_{St} = V^0_{St}\,(\delta_R u)^{\gamma(St(R))}\,\left[(\delta_R u)^2 + c_1\,(\delta u^*(St))^2\right]^{[1-\gamma(St(R))]/2}. \tag{3.6}$$

The above expression is a Batchelor-like parametrization but in the velocity space, with a transient velocity given by the Stokes velocity, $\delta u^*(St)$; the adimensional constant c_1 is the only free parameter—order unity, in our data—once the large-scale normalization function V^0_{St}, the caustic exponent $\gamma(x)$ and the reference fluid velocity increment $\delta_R u$ are given.

In figure 3, we show the result of the fit in terms of the expression (3.6), where the caustics scaling exponent has been chosen, as mentioned above, equal to $\gamma(x) = [1 - 2/\pi\ \arctan(g_1\,x)]$; this functional form with $g_1 = 1.2$ provides a good fit to the numerical results.

The qualitative trend is very well captured by the interpolation function proposed. Note that in expression (3.6), the argument of $\gamma(St)$ is not the simple Stokes number at the Kolmogorov scale, but the scale-dependent one $St(R)$: $\gamma(St) \rightarrow \gamma(St(R))$. This

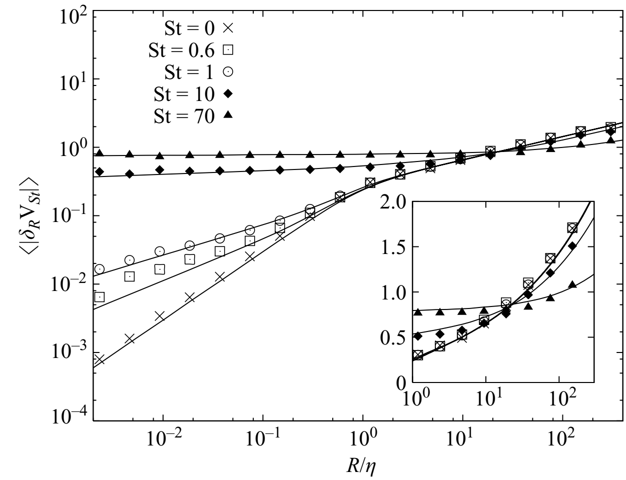

FIGURE 3. Scaling behaviour of the particle velocity structure function of order one vs. the normalized scale R/η. Solid lines indicate the fit of the data of figure 2, Run II, using the interpolation formula (3.6). Here, the large-scale prefactors V_{St}^0 are those measured in Run II of the simulation and shown in the inset of figure 2(*b*). (Inset) Enlargement of the crossover range, where $\delta_R u \sim \delta_R V$.

further ingredient is needed to take into account the fact that in the presence of a rough underlying fluid velocity, as it happens in the inertial range of scales, no simple power-law behaviour is expected for the scaling of particle velocity statistics. This was remarked by Bec *et al.* (2008), in the study of heavy particle turbulent dispersion in random flows.

Equation (3.6) clearly matches the two limiting behaviours for very small and very large separations. In the former case, inertia dominates the small-scale velocity statistics with respect to the underlying smooth fluid velocity and caustics lead to a pure power-law behaviour

$$\delta_R V \sim V_{St}^0 (\delta_R u)^{\gamma(St)} \sim U_0 V_{St}^0 \left(\frac{R}{L}\right)^{\gamma(St)}, \quad R \ll \eta, \tag{3.7}$$

where the local Stokes number has attained its dissipative limit $St(R) \rightarrow St$.

In the latter case, at very large scales $R \gg R^*(St)$, inertia is subleading, and the typical velocity difference between particles is close to the fluid velocity increment

$$\delta_R V_{St} \sim V_{St}^0\, \delta_R u, \quad \eta \ll R^*(St) \ll R. \tag{3.8}$$

At intermediate scale, for large Stokes, $St \geqslant 1$, inertia brings in a non-trivial dependency via the scale Stokes number, $St(R)$, and we expect a pseudo-power-law scaling:

$$\delta_R V_{St} \sim (\delta_R u)^{\gamma(St(R))} \sim R^{\gamma(St(R))/3}, \quad \eta \ll R \ll R^*(St). \tag{3.9}$$

Summarizing, we propose that by changing the Stokes and Reynolds numbers, different regimes governing the particle velocity statistics can be distinguished. The relevance of such regimes of the particle velocity statistics for the associate relative dispersion dynamics can be easily explained with the help of the sketch shown in figure 4. In the parameter space of inertia and scale separation (St, R), we can distinguish three regions depending on whether inertia is strong or weak, and whether

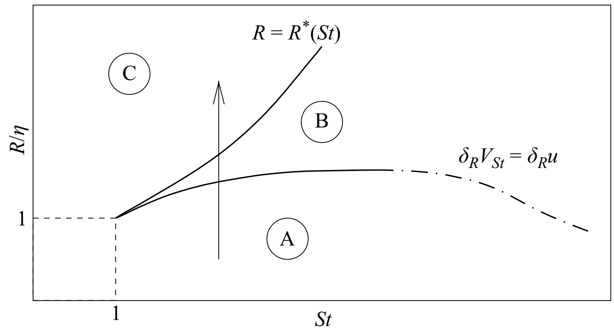

FIGURE 4. Sketch of the different regimes for heavy pairs expected in the parameter space of inertia St and scale separation R. The curve $St(R^*) = 1$ separates the region of low inertia $St(R) \leqslant 1$, region (C), from two regions (A) and (B), where inertia is important $St(R) \geqslant 1$. In latter case, we can distinguish the dispersion regime when inertia is important and particle velocity difference is larger than the corresponding fluid one at the same scale: this is region (A). Besides, there is an intermediate regime in which inertia still affects relative dispersion, but particle velocity difference is smaller than the corresponding fluid one: this happens in region (B). The borderline between regions (A) and (B) is given by the curve $\delta_R V_{St} = \delta_R u$. The relative dispersion of heavy pairs of Stokes St with initial separation R can be described as starting from the corresponding position in this plane and then evolving upwards along the vertical arrow.

particle velocity difference is large or not with respect to the fluid velocity difference at comparable scale. In agreement with what commented before, we pose that the curve $St(R^*) = 1$ distinguishes the region of weak ($St(R) \leqslant 1$) and strong inertia ($St(R) \geqslant 1$).

According to figure 4, in regime (A) inertia is important since the scale $R^*(St) \gg \eta$ and, moreover, the typical particle velocity increments are larger than the fluid increments. In region (B), inertia is still important but particle velocity increments are depleted with respect to the fluid increments. This typically happens for large Stokes numbers, and in our DNS it is visible only for very large separations $R(t)$ of the highest Stokes $St = 70$. Finally, regime (C) is characterized by a weak inertia, which appears only in the filtering factor for the velocity large-scale amplitude and possibly in subleading corrections to the tracers' relative dispersion.

Even for the largest value of the Reynolds and Stokes numbers achieved in our DNS, it is very difficult to disentangle quantitatively the above-mentioned regimes because of the closeness of the three relevant scales, η, $R^*(St)$ and L. Still, the quality of the fit shown in figure 3 using the global functional dependence given by (3.6) makes us confident that the main physical features are correctly captured. Before closing this section, we note that there is no reason to assume that the functional form entering in the pseudo-power-law scaling in the inertial range, $\gamma(St(R))$, in (3.9) is equal the one characterizing the scaling in the viscous range, $\gamma(St)$, in (3.7). Hints for this observation come from results obtained by Bec *et al.* (2008) for random flows, where a very high statistical accuracy can be achieved; there, depending on if the underlying fluid velocity is spatially smooth or rough, a slightly different functional form has been found.

The previous analysis gives us a clear quantitative picture of the scale and velocity ranges where caustics play a role in the particle dynamics. For example, for moderate Stokes numbers, we have important departure from the tracers statistics only for very small scales, i.e. caustics gives a singular contributions to the particle velocity increments inside the viscous range; then, at larger scales, the particle velocity scaling becomes indistinguishable from the tracer velocities. Clearly, for such Stokes number, no important corrections for particle separation evolution are expected with respect to the usual Richardson dispersion observed for tracers. This is because particle pairs tend to separate, and very soon all pairs will attain separations where their velocities are very close to the underlying fluid. On the other hand, for very heavy particles, those with Stokes time falling inside the inertial range of fluid velocity statistics, the contribution from the caustics will also be felt at relatively large scales, up to $R \sim R^*(St)$. Pair separations attain such scales when the initially large relative velocity difference has relaxed and become smaller than the corresponding fluid one—crossing from regions (A) to (B). Note that at $R \sim R^*(St)$, we have that $\delta_{R^*} V_{St} \simeq V^0_{St} \delta_{R^*} u$, i.e. there is a non-trivial effect from inertia. Moreover, for large Stokes, at scales $R > R^*(St)$, particle velocity increments are smaller than the fluid counterparts, indicating an important depletion induced by the Stokes drag on the particle evolution.

It is clear from the above discussion that new physics should appear for the value of inertia and scales separation of region (B). This regime—which we cannot access with the present data — is the one where a new law of pair separation should appear as recently suggested by Fouxon & Horvai (2008). A discussion of the dispersion regimes of inertial particle pairs follows in the next section in terms of the time behaviour of the mean-square separation distance.

4. Dispersion regimes and corrections due to inertia

In this section, we analyse the effects of inertia on the mean-square separation of heavy particle pairs with a given initial separation distance, R_0 at time $t = t_0$, as a function of the Stokes number:

$$\langle (R(t))^2 \mid R_0, t_0 \rangle_{St} = \langle |\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)|^2 \rangle_{St}, \tag{4.1}$$

where in the left-hand side the average is performed over all pairs of particles such that $|\boldsymbol{X}_1(t_0) - \boldsymbol{X}_2(t_0)| = R_0$. The study of the relative dispersion of small, neutrally buoyant tracer particles has recently been the subject of renewed interest. This has been motivated by the fact that very accurate data—highly resolved in time and space—have become available experimentally (Ott & Mann 2000; Bourgoin *et al.* 2006) and numerically (Yeung & Borgas 2004; Biferale *et al.* 2005, 2006). These studies have confirmed and assessed the extent of validity of what was predicted by Richardson (1926) and Batchelor (1952), i.e. the existence of different dispersive regimes for tracer pairs in turbulent flows, depending on the value of their initial distance and on the time scale considered.

When released in a statistically homogeneous and isotropic turbulent flow, with an initial separation R_0 in the inertial range of scales for fluid velocity, i.e. $\eta \simeq R_0 \ll L$, tracer pairs initially separate according so the so-called Batchelor regime

$$\langle (\boldsymbol{R}(t) - \boldsymbol{R}_0(t_0))^2 \rangle_{St=0} = \tilde{C} (\varepsilon R_0)^{2/3} t^2, \quad \tau_\eta \ll (t - t_0) \ll t_B,$$

or equivalently,

$$\langle (R(t))^2 | R_0, t_0 \rangle_{St=0} \simeq R_0^2 + \tilde{C} (\varepsilon R_0)^{2/3} t^2, \quad \tau_\eta \ll (t - t_0) \ll t_B, \tag{4.2}$$

where $\tilde{C}$ is supposed to be a universal constant. This *ballistic* regime appears because tracers initially separate *as if* the underlying velocity field was frozen, and it lasts for a time scale that is a function of the initial separation itself, $t_B = (R_0^2/\varepsilon)^{1/3}$

(see Batchelor 1952; Sawford 2001; Bourgoin *et al.* 2006). After such a transient initial time, the relative separation dynamics forgets the initial conditions and tracers separate explosively with a power-law behaviour given by the Richardson law:

$$\langle (R(t))^2 | R_0, t_0 \rangle_{(St=0)} = g\,\varepsilon\,t^3, \quad t_B \ll (t - t_0) \ll T_L, \tag{4.3}$$

where g is known as the Richardson constant, and T_L is the eddy turnover time at the integral scale L. As set out by Monin & Yaglom (2007), the tracer separation PDF—which will be discussed later—has a similar scaling behaviour in these ranges.

Note that a Taylor expansion at short times of the relative dispersion $\langle (\boldsymbol{R}(t) - \boldsymbol{R}_0(t_0))^2 \rangle_{St=0}$ would also lead to the appearance of a cubic in time term, but with negative sign, i.e. $\propto -t^3$. This, however, has nothing to do with the Richardson superdiffusive regime, which pertains to an asymptotic non-differentiable regime in the particle separation that is reached at long time. Richardson regime is obtained by a simple dimensional argument which makes use of the Kolmogorov similarity law for the Eulerian velocity increments in a turbulent flow.

A remarkable aspect of the Richardson separation law in the inertial range is its universality. Note the absence of initial separation dependence, an effect also dubbed *intrinsic stochasticity* (Weinan & Vanden Eijnden 2000), which is just the signature of the non-Lipschitz nature of the velocity field driving the separation between tracers, when their mutual distance is in the inertial range of fluid velocity statistics. The experimental and numerical validation of the previous prediction (4.3) has proved to be particularly difficult in laboratory or numerical flows (see e.g. the discussion of Sawford, Yeung & Hackl 2008), the main reason being the strong effects due to viscous and large scales in the tracers' dynamics. Measurements in flows at larger Reynolds numbers and with pairs having smaller initial separations would be required. To overcome these limitations within the present available setups, a series of techniques have been developed, including the study of *doubling time statistics*, i.e. the probability distribution function of the time needed for a pair to double its separation (Boffetta & Sokolov 2002). Thanks to these methods, a fairly good agreement on the value of the Richardson constant has been achieved in numerical data (see e.g. Biferale *et al.* 2005). Here, we want to study how the tracer behaviour is modified by the presence of small-scale caustics in particular and by inertia effects in general, for the case of heavy particle pairs. Standard direct measurements of the moments of separation as a function of time will be considered, while application of doubling time statistics is left for future studies.

In figure 5, we show the behaviour of the mean-square separation at varying the Stokes number, and for two values of the initial distance. We start with data at the lowest resolution, i.e. Run I at $Re_\lambda \simeq 200$, and for moderate Stokes numbers, $St \sim \mathcal{O}(1)$. Initial distances are chosen equal to $R_0 \leqslant \eta$ (a, c) and $R_0 \in [4:6]\,\eta$ (b, d).

If the initial distance is small enough (a, c), the presence of caustics in the particle velocity field at initial time gives a very remarkable departure from the tracer behaviour. At increasing the Stokes number, such departure is more and more evident, and it lasts for a time lag that becomes longer and longer. For the highest value of the Stokes number shown in figure 5(a, c), $(St = 3.3)$, a sensible difference from the tracer behaviour is observed over almost two decades: $t \in [0.1:10]\,\tau_\eta$. A way to better visualize the departure from the tracer statistics consists in plotting the mean-square separation for heavy pairs of different Stokes numbers, normalized to

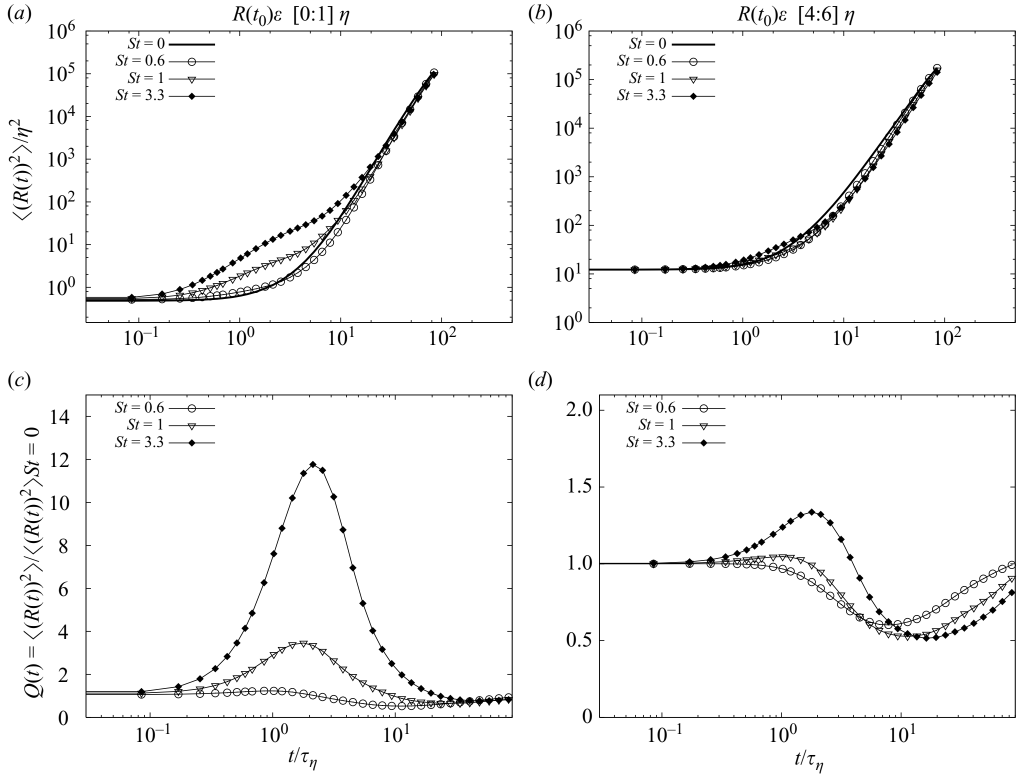

FIGURE 5. (a, b) The mean-square separation vs. time, for heavy particles at changing St and the initial distance R_0. Time is normalized by the Kolmogorov time scale τ_η. (a, c) Results for $St = 0, 0.6, 1$ and 3.3; initial distance $R_0 \in [0 : 1]\eta$, Run I. Error bars due to statistical fluctuations are of the order of the symbol size. Note that the two largest Stokes numbers show a time lag interval where separation proceeds faster than tracers. (b, d) Mean-square separation vs. time for pairs with initial separation $R_0 \in [4 : 6]\eta$. Stokes numbers are the same as in (a, c). Note that now only the dispersion of particle pairs with $St = 3.3$ exhibits a small departure from the underlying fluid. For the smaller Stokes, typical length scale of caustic-like velocity increment is smaller than the initial separation R_0, and particle pairs, therefore, separate as fluid tracers do. (c, d) The ratio $Q(t)$, between the heavy particle separation and the tracer data, vs. time and for $St = 0.6, 1, 3.3$, for the two selected initial separations. Symbols are the same as for (a, b).

the tracer one,

$$Q(t) = \frac{\langle (R(t))^2 \rangle_{St}}{\langle (R(t))^2 \rangle_{(St=0)}}. \tag{4.4}$$

This quantity is shown in figure 5(c, d). For heavy pairs starting at $R_0 \simeq \eta$ and with $St = 3.3$, the relative difference is as large as 10 at its maximum for $t \sim \tau_\eta$. However, such effect becomes progressively less important if we start the separation experiment from larger initial distances as shown in figure 5(b, d). This is because, at these same Stokes numbers, the deviation of particle velocity difference with respect to the underlying fluid, due to caustics, has already decreased. This is equivalent to state that, for these Stokes numbers, the typical length scale of caustic-like velocity increment is smaller than the initial separation R_0, and particle pairs therefore separate as fluid tracers do. At larger time lags, whatever the value of the initial separation, the Richardson dispersion regime is recovered.

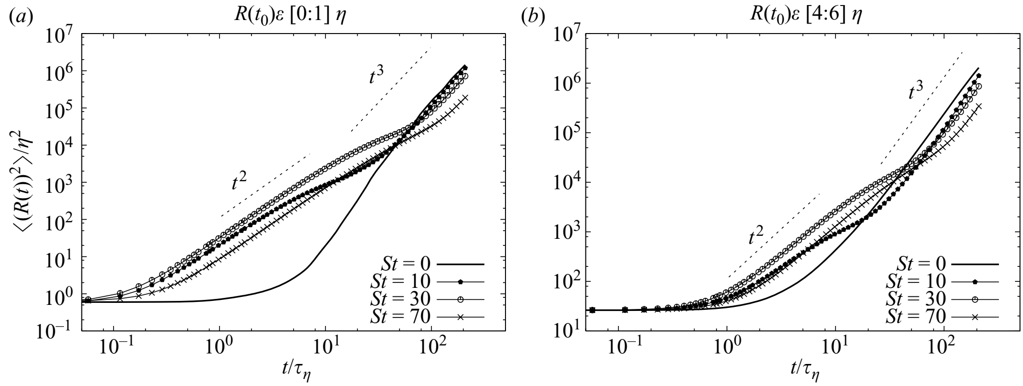

FIGURE 6. Mean-square separations vs. time for pairs with $St = 10, 30, 70$, at $Re_\lambda = 400$. Two initial distances $R_0 \in [0:1]\eta$ (*a*) and $R_0 \in [4:6]\eta$ (*b*) Error bars due to statistical fluctuations are of the order of the symbols size. Tracers (solid lines) are also shown for comparison. Note the ballistic behaviour for the heavy particle separation observed in the *caustics dominated* time interval. For very large time lags, a Richardson-like behaviour starts to develop but with a less intense overall speed of separation, because of the depletion effects of the $V_{St}^{(0)}$ prefactor in the particle velocity increments for large Stokes numbers. The slopes of the Batchelor, $\langle R^2(t)\rangle \propto t^2$, and Richardson, $\langle R^2(t)\rangle \propto t^3$, dispersion regimes has also been drawn for reference.

We now consider what happens for larger Stokes numbers. In figure 6, we show the results for the mean-square separation at $St = 10$, 30 and 70 and for the large Reynolds number, $Re_\lambda \simeq 400$. Both initial distances, $R_0 \in [0:1]\eta$ and $R_0 \in [4:6]\eta$, are displayed. As one can see, for the large value equal to $St = 70$, the tracer-like behaviour is never recovered, and even the separation of pairs starting with the largest distance R_0 is affected. The transient regime dominated by the caustics invades the whole inertial range. Since particle pairs need a very long time to decrease their initial velocity difference to the value of the fluid increment at the corresponding scale, they separate with a quasi-ballistic behaviour: $\langle R^2(t)\rangle_{St} \propto t^2$.

The above scenario can be interpreted in terms of caustic *range of scales*. At any value of the inertia, there exist a spatial length, of the order of the scale $R^*(St)$, which identifies the typical spatial size of caustics, i.e. the range of scales where particle velocity increments are uncorrelated from the underlying fluid velocity field. If the initial pair separation R_0 is taken inside this region (figure 5*a*, *c*), particle pair separation starts much faster than for fluid tracers because of the much more intense velocity differences felt by the pairs inside the caustics. When particle pairs reach a separation larger than $R^*(t)$, they start to be synchronized with the underlying fluid velocity, recovering the typical Richardson dispersion. However, if the initial separation is larger than the caustics' typical scale, the evolution of inertial particle pairs is almost indistinguishable from the tracers. Finally, whether or not a Richardson-like behaviour is recovered for very large inertia may also depend on the Reynolds number. In the limit of larger and larger Reynolds, at fixed Stokes number, one may expect a final recovery of the fluid tracers behaviour even for very heavy particle pairs.

4.1. *Mean-field approach to heavy particle dispersion*

The turbulent relative dispersion of fluid tracers can be easily modelled by applying K41 scaling theory for isotropic and homogeneous flows to the fluid velocity increments governing particle separation dynamics (see, e.g. Ouellette *et al.* 2006).

Indeed, if $\boldsymbol{R}(t)$ is the tracer separation vector at a given time, its evolution is completely specified by

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{u}(\boldsymbol{X}_1, t) - \boldsymbol{u}(\boldsymbol{X}_2, t) = \delta_R \boldsymbol{u}(\boldsymbol{R}, t), \tag{4.5}$$

together with the initial condition $\boldsymbol{R}(t_0) = \boldsymbol{R}_0$. Hence, we can directly write an equation for the root-mean-square (r.m.s.) separation $r(t) \equiv \langle |\boldsymbol{R}(t)|^2 \,|\, R_0, t_0\rangle^{1/2}$

$$\dot{r} = \frac{1}{r}\langle \boldsymbol{R}(t) \cdot \delta_R \boldsymbol{u}(\boldsymbol{R}(t), t) \,|\, R_0, t_0\rangle, \quad \text{with } r(t_0) = R_0. \tag{4.6}$$

Here, for the sake of simplicity, we have omitted the subscript $\langle \cdot \rangle_{St}$ to distinguish the tracers case from the heavy particles one.

The exact form of the correlations among tracers relative separation and the underlying fluid velocities is clearly out of control, and phenomenological closures have to be formulated to solve equations such as the previous one. A zero-order approximation is to factorize correlations and assume the following mean-field closure for the right-hand side of (4.6):

$$\langle \boldsymbol{R}(t) \cdot \delta_R \boldsymbol{u}(\boldsymbol{R}(t), t) \,|\, R_0, t_0\rangle \approx \langle \boldsymbol{R}^2 \,|\, R_0, t_0\rangle^{1/2} \langle \hat{\boldsymbol{R}} \cdot \delta_R \boldsymbol{u}\rangle = C\, r\, S_{2//}^{1/2}(r), \tag{4.7}$$

where $S_{2//}(r)$ is the second-order Eulerian longitudinal structure function of the underlying homogeneous and isotropic turbulent flow, and C is an order-unity positive dimensionless constant. The above closure is qualitatively well reproduced by the DNS data (not shown). According to K41 phenomenology, this structure function behaves in the inertial range as $S_{2\parallel}(r) \propto (\varepsilon\, r)^{2/3}$. The closure finally leads to

$$\dot{r} = C\, \varepsilon^{1/3} r^{1/3}, \quad \text{so that} \quad r(t) = \left[R_0^{2/3} + (2C/3)\, \varepsilon^{1/3}(t - t_0)\right]^{3/2}. \tag{4.8}$$

Such an approximation gives a complete qualitative picture of the time evolution of the mean-square separation between tracers. In particular, it encompasses the two important regimes of relative dispersion. On the one hand, when $(t - t_0) \ll t_B = (R_0^2/\varepsilon)^{1/3}$, a Taylor expansion of the solution (4.8), valid at short times only, gives the Batchelor regime $r(t) \simeq R_0 + C\,(\varepsilon R_0)^{1/3}\,(t - t_0)$. On the other hand, for large times, $(t - t_0) \gg t_B$, it reproduces Richardson's law $r(t) \simeq (2C/3)^{3/2}\,(\varepsilon\, t^3)^{1/2}$.

In the case of inertial particles, the number of degrees of freedom to describe the dynamics has obviously been increased. The separation between two heavy particles obeys

$$\ddot{\boldsymbol{R}}(t) = -\frac{1}{\tau_s}\left[\dot{\boldsymbol{R}}(t) - \delta_R \boldsymbol{u}(\boldsymbol{R}, t)\right]. \tag{4.9}$$

In order to derive *mean-field* equations, one has to track simultaneously the average distance and velocity difference between particles. Following the same pattern as for tracers, we introduce the particle velocity structure function $v(t) \equiv \langle |\delta_R \boldsymbol{V}(t)|^2 \,|\, R_0, t_0\rangle^{1/2}$, where $\delta_R \boldsymbol{V}(t) = \dot{\boldsymbol{R}}(t)$ is the velocity difference between the pair particles. One can proceed, as previously, to write from (4.9) exact equations for $r(t)$ and $v(t)$:

$$\ddot{r} = \frac{1}{r}(v^2 - \dot{r}^2) - \frac{1}{\tau_s}\left[\dot{r} - \frac{1}{r}\langle \boldsymbol{R} \cdot \delta_R \boldsymbol{u}\rangle\right], \tag{4.10}$$

$$\dot{v} = -\frac{1}{\tau_s}\left[v - \frac{1}{v}\langle \delta_R \boldsymbol{V} \cdot \delta_R \boldsymbol{u}\rangle\right], \tag{4.11}$$

where, for the sake of a lighter notation, the indication of conditional ensemble averages was dropped. It is worth noting that the r.m.s. velocity difference $v(t)$ evolves with a dynamics that resembles closely that of heavy particles. However, $v(t)$ does not coincide with the time derivative of the mean distance $r(t)$. It is thus useful to rewrite the above equations introducing a sort of transverse particle velocity component w defined as

$$\dot{r} = v - w. \tag{4.12}$$

We can also write an exact equation for the evolution of w

$$\dot{w} = -\frac{1}{\tau_s} w - (2v - w)\frac{w}{r} - \frac{1}{\tau_s}\left[\frac{1}{r}\langle \boldsymbol{R} \cdot \delta_R \boldsymbol{u}\rangle - \frac{1}{v}\langle \delta_R \boldsymbol{V} \cdot \delta_R \boldsymbol{u}\rangle\right]. \tag{4.13}$$

Of course, the exact equations (4.11)–(4.13) are not closed without supplying the correlation between the particle evolution and the underlying fluid. As in the case of tracers, the first unclosed term appearing in the right-hand side of (4.13) is approximated by (4.7). Next, the term involving the correlation between fluid and particle velocity differences is again obtained by splitting correlations and is approximated by

$$\langle \delta_R \boldsymbol{V} \cdot \delta_R \boldsymbol{u}\rangle \approx \langle |\delta_R \boldsymbol{V}|^2\rangle^{1/2} \langle |\delta_R \boldsymbol{u}|^2\rangle^{1/2} = C_1\, v\, S_2^{1/2}(r), \tag{4.14}$$

where $S_2(r)$ denotes the full second-order structure function of the fluid velocity field, and C_1 is a non-negative order-unity dimensionless constant. When r is in the inertial range, K41 phenomenology implies that $S_2(r) \propto (\varepsilon r)^{2/3}$. Finally, these approximations lead to a closed set of equations for the time evolution of the average separation and velocities r, v and w

$$\dot{r} = v - w, \tag{4.15}$$

$$\dot{v} = \frac{1}{\tau_s}[C\, \varepsilon^{1/3} r^{1/3} - v], \tag{4.16}$$

$$\dot{w} = -\frac{1}{\tau_s} w - (2v - w)\, \frac{w}{r} + \frac{1}{\tau_s} B\, \varepsilon^{1/3} r^{1/3}, \tag{4.17}$$

where the positive dimensionless constants B and C reflect the lack of control on the prefactors of the scaling laws in the closures (4.7) and (4.14). This system of equations is supplemented by the initial conditions $r(t_0) = R_0$, $v(t_0) = \langle |\delta_{R_0} \boldsymbol{V}|^2\rangle^{1/2}$ and $w(t_0) = v(t_0) - \langle \boldsymbol{R}_0 \cdot \delta_{R_0} \boldsymbol{V}\rangle / R_0$, which clearly depend on the dispersion experiment under consideration. It is worth noting that this system of equations reduces to the mean-field equation (4.8) for tracers in the limit of vanishing inertia $\tau_s \to 0$.

Similar to the case of tracers, the *crude* approximations of the evolution of the r.m.s. distance between heavy particles (4.15)–(4.17) are able to capture the main features of the separation time behaviour. In figure 7, we show the result of the numerical integration of the set of equations (4.15)–(4.17), together with DNS data from Run II, for two different large values of the Stokes number, namely $St = 10$ and $St = 70$. Note that for the numerical integration of the mean-field closure, the following appropriate choice was adopted for the dimensionless constants: $B = 0$ and $C = 2$. In particular, the former is motivated by the fact that in the limit of very small Stokes numbers, the two terms in the square brackets of (4.13) display the same scaling behaviour and differ only by a constant factor smaller than one (not shown). For intermediate and large Stokes, deviations occur particularly in the dissipative range of scales. However, this does not affect much the quality of the approximation. The coefficient C is to be chosen positive and $O(1)$, since already in the tracers limit there is a

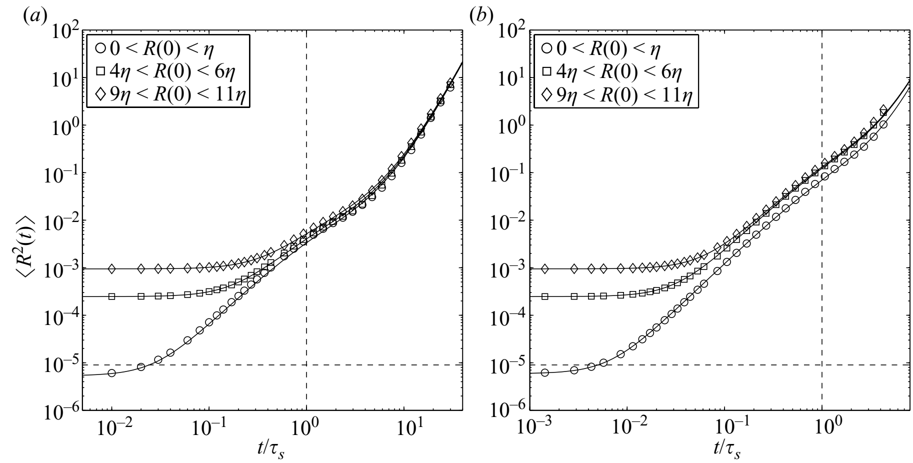

FIGURE 7. Mean-square separation vs. time for Run II and Stokes numbers $St = 10$ (*a*) and $St = 70$ (*b*). Data are shown for the available choices of the initial separations : $R_0 \leqslant \eta$, $R_0 \in [4:6]\,\eta$ and $R_0 \in [9:11]\,\eta$ (from bottom to top). Symbols refer to DNS data, while solid lines are mean-field solutions to (4.15–4.17), with the dimensional prefactors set as $B = 0$ and $C \simeq 2$. Note that time is made adimensional with the Stokes time τ_s. Vertical dashed lines mark $t/\tau_s = 1$.

non-zero correlation between the separation and the fluid velocity increment, due to the conditioning with respect to the initial separation in the statistics.

For fixed initial separation and at increasing the intensity of the caustic-like velocity increments in the initial condition (i.e. at increasing inertia), the transient deviation from the Richardson behaviour becomes more and more evident at intermediate times (of the order of the Stokes time τ_s, not shown). Clearly, the simple approach proposed can be valid only in a limited region of the phase space, where the initial conditions are, at least, at the edge of the inertial range so that K41 scaling is correct for the fluid velocity second-order increments. Moreover, the matching scale where particle velocity increments become of the order of the fluid increments has to fall in the inertial range too; if this is not the case, then pairs enter the regime where inertia is important but particle relative velocity is small at scales of the dissipative range, and the mean-field closure proposed above becomes inadequate. A detailed quantitative comparison of the realm of applicability of the mean-field approach—including effects of viscous scales and small-scale caustics—will be the object of future work.

4.2. *Cross-over between relaxation to the fluid velocity and Richardson behaviour*

Despite its simplicity, the mean-field approach described above is able to correctly reproduce the pair dispersion of heavy particles with initial data in the inertial range. We might wonder if one can draw an even simpler qualitative picture of pair dispersion. For this, we consider the behaviour of particle pairs with moderately large Stokes numbers, for which inertia plays an important role for the initial transient, and the Richardson behaviour is slowly recovered well inside the inertial range of scales. For simplicity, we assume that the scale where the fluid and particle velocity becomes of the same order, $\delta_R \boldsymbol{V} \sim \delta_R \boldsymbol{u}$, and the scale $R^*(St)$, where inertia ceases to be important, are very close. As it is clear from the sketch of figure 4, this may not always be the case because of the effect of the normalization factor V^0_{St} for large Stokes; in the figure, it corresponds to Stokes numbers with a narrow transient region (B).

The general picture then goes as follows. Initially, particles separate almost ballistically during a time which is of the order of (or larger than) the time needed by their initial, caustics-dominated, velocities to relax to the fluid velocity. After that time, particles behave as tracers and reconcile with a standard Richardson dispersion. This is a first-order approximation since (i) the fluid flow actually correlates to the particle dynamics already at very small times, and (ii) inertia effects are present up to large times as discussed above. Nevertheless, such an approximation should give the two correct qualitative asymptotic behaviours, at small and large time scales. Since we consider moderately large values of the Stokes number, the initial typical particle velocity can be assumed to be much larger than the fluid velocity, i.e. $|\delta_R \boldsymbol{V}| \gg |\delta_R \boldsymbol{u}|$. Under these hypotheses, there is an initial time interval during which the difference between particle velocities obeys $\delta_R \dot{\boldsymbol{V}} \approx -(\delta_R \boldsymbol{V})/\tau_s$ (see (4.11)), and thus $\delta_R \boldsymbol{V}(t) \simeq (\delta_R \boldsymbol{V}(t_0))\, \mathrm{e}^{-(t-t_0)/\tau_s}$. As a consequence, the mean-square separation between particles evolves initially as

$$\langle |\boldsymbol{R}^2(t)| \,|\, R_0, t_0\rangle = R_0^2 + 2\tau_s \langle \boldsymbol{R}(t_0) \cdot \delta_R \boldsymbol{V}(t_0)\rangle (1 - \mathrm{e}^{-(t-t_0)/\tau_s}) + \tau_s^2 \,\langle (\delta_R \boldsymbol{V}(t_0))^2\rangle (1 - \mathrm{e}^{-(t-t_0)/\tau_s})^2. \quad (4.18)$$

This should be approximately valid up to a time scale, in the inertial range, where $|\delta_R \boldsymbol{V}| \sim |\delta_R \boldsymbol{u}| \sim (\varepsilon R)^{1/3}$; it is easy to show that such a time scale is proportional to the particle response time τ_s. For larger times, inertia effects become subdominant and heavy pair dispersion suddenly gets synchronized to a Richardson-like regime. Nevertheless, this Richardson regime has started only after the previous relaxation has ended, that is at a distance much larger than the original separation R_0 of the particle pair. The combination of this initial exponential relaxation of heavy particles with moderately large inertia, plus the later standard Richardson diffusion, are the two main features due to inertia in the inertial pair dispersion. This is indeed confirmed by figure (8), where we compare DNS data for mean-square separation, with the two phenomenological regimes just described, for which we have assumed that $\langle \boldsymbol{R} \cdot \delta_R \boldsymbol{V}(t_0)\rangle \simeq 0$. As we can see, the main qualitative trends of the small and large time behaviours are very well captured. Let us conclude this section by mentioning that further refinements of the mean-field approach here proposed would require the inclusions of viscous behaviour in the closures (4.16–4.17) when the initial particle separation is taken well inside the viscous range. We also note that an attempt of modelling heavy particles relative dispersion in a similar direction was performed by El Maihy & Nicolleau (2005).

4.3. *Subleading terms in the Richardson regime*

As seen in the above subsections, the most noticeable effect of inertia on the mean pair dispersion is a long transient regime that takes place before reaching a Richardson explosive separation (4.3), and that this regime is due to the relaxation of particle velocities to those of the fluid. As we now argue, at larger times—corresponding to regime (C)—there is still an effect of particle inertia that can be measured in terms of subleading corrections to the Richardson law. To estimate these corrections, let us assume that in the mean-field equation (4.16), the term stemming from the fluid velocity $C\varepsilon^{1/3} r^{1/3}$ is much larger than the inertia term $\tau_s \dot{v}$. This is true when $St(r) \ll 1$, i.e. at times t when $r(t) \gg R^*(St)$. In this asymptotic, one can infer that the transverse velocity component w is much smaller than the total velocity v, so that $\dot{r} \simeq v$ (see (4.12)). In the spirit of the weak inertia expansion derived by Maxey (1987),

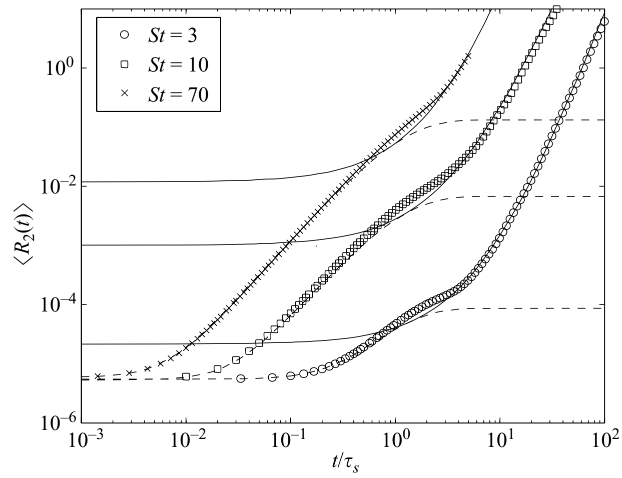

FIGURE 8. Mean-square separation vs. time from DNS data of Run II, for three different values of the Stokes numbers. Note that time is normalized with the Stokes time τ_s. Solid lines represent the initial, almost ballistic evolution due to the exponential relaxation of velocity statistics, and the dashed lines correspond to the Richardson regime. With a suitable tuning of the free parameters, here the Richardson constant g and the initial velocity increment value $\langle(\delta_R V(t_0))^2\rangle$, both temporal behaviours are reproduced.

we next write a Taylor expansion of (4.16) to obtain

$$\dot{r} \approx v \approx C\varepsilon^{1/3}r^{1/3} - \tau_s\langle|(\mathrm{d}/\mathrm{d}t)\,\delta_r\boldsymbol{u}|^2\rangle^{1/2} \approx C\varepsilon^{1/3}r^{1/3} - \tau_s\langle|\delta_r\boldsymbol{a}|^2\rangle^{1/2}, \tag{4.19}$$

where $\delta_r\boldsymbol{a} = \delta_r(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u})$ denotes the increment of the fluid acceleration over the separation r. Next, we assume scaling invariance of the turbulent acceleration field, that is, according to dimensional arguments of K41 theory, $|\delta_r\boldsymbol{a}| \sim \varepsilon^{2/3}r^{-1/3}$. Equation (4.19) can then be rewritten as

$$\dot{r} = C\varepsilon^{1/3}r^{1/3}\left(1 - A\,\tau_s\,\varepsilon^{1/3}r^{-2/3}\right) = C\varepsilon^{1/3}r^{1/3}\left(1 - A\,St(r)\right), \tag{4.20}$$

where A is an order-unity constant. The initial condition is given by $r(t_0) = r_0$, where the initial separation has to be chosen such that $St(r_0) \ll 1$. We can next integrate the approximate dynamics perturbatively in terms of the small parameter $St(r_0)$ by expanding the separation as $r(t) = \rho_0(t) + \rho_1(t) + \rho_2(t) + \cdots$. The leading order is $\rho_0(t) = [r_0^{2/3} + (2C/3)\,\varepsilon^{1/3}t]^{3/2}$ and corresponds to the relative dispersion of a pair of tracers. The first-order correction is $\rho_1(t) = -\,\tau_s\,\varepsilon^{1/3}A\,\ln(\rho_0(t)/r_0)\,\rho_0^{1/3}(t)$.

At times much larger than the Batchelor time associated with the initial separation r_0, i.e. for $t \gg (r_0^2/\varepsilon)^{1/3}$, the leading term follows the Richardson explosive law $\rho_0(t) \simeq (2C/3)^{3/2}(\varepsilon\,t^3)^{1/2}$. This finally implies that in the asymptotics $t \gg (r_0^2/\varepsilon)^{1/3} \gg \tau_s$, one can write

$$r^2(t) \propto g\,\varepsilon\,t^3\,\left[1 - D\,(t/\tau_s)^{-1}\ln(t/\tau_s)\right], \tag{4.21}$$

where g is the Richardson constant introduced in §4 and D is an order-unity factor, which *a priori* does not depend either on the particle Stokes number or on the initial particles separation.

This behaviour is confirmed numerically. Figure 9 shows the long-time dependence of the ratio $Q(t)$ between the mean-square separation of heavy particles and that of tracers as defined by (4.4). It is clear that data almost collapse on a line $\propto 1/t$, confirming the behaviour (4.21) predicted above. The small-inertia expansion

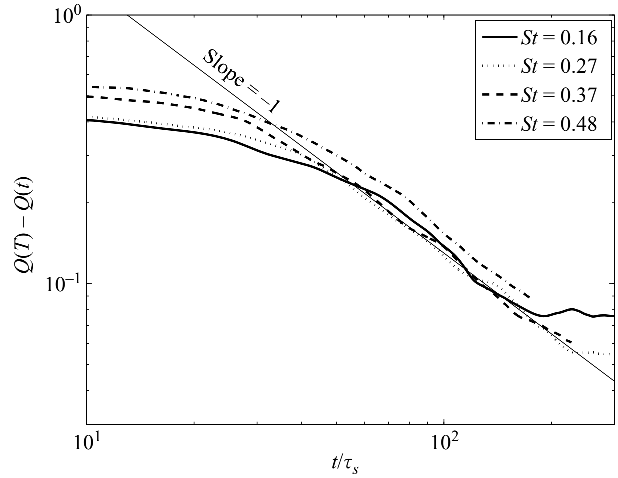

FIGURE 9. Large-time behaviour of the mean-square separation normalized to that of tracers as defined from (4.4) for Run I and various values of the Stokes number as labelled. Deviations from the fluid tracer Richardson law behave as $(t/\tau_s)^{-1}$. The limiting value $Q(T)$ has been chosen different from unity as effects of inertia are still present at the largest scale of the flow.

à la Maxey should be valid when the particle response time is much shorter than the turnover time associated with the distance between the particles. Because of the moderate values of the Reynolds number and the resulting lack of time scale separation, there are strong limitations on the values of the Stokes number for which a $1/t$ behaviour can be observed. Note, finally, that only results from Run I are displayed in figure 9. The reason is that large time statistics of tracer dispersion in Run II is not as well statistically converged, leading to more noisy data. The qualitative picture is, however, very similar, in the same range of Stokes numbers.

To conclude, let us stress that, in the derivation of (4.20) from (4.19), we have assumed the K41 scaling to hold for the acceleration field, and thus for the pressure gradient. However, it is well known that the scaling properties of pressure field are still unclear: they might depend on the turbulent flow Reynolds number and/or on the type of flow (see, e.g. Gotoh & Fukayama 2001; Xu *et al.* 2007). As stated by Bec *et al.* (2007*a*), rather than being dominated by K41 scaling, numerically estimated pressure increments of Run I ($Re_\lambda \simeq 200$) seem to be ruled by sweeping, so that $|\delta_r \boldsymbol{a}| \sim U_0\, \varepsilon^{1/3} r^{-2/3}$. One can easily check that this difference in scaling leads to a behaviour similar to (4.21), except that this time logarithmic corrections are absent and that the non-dimensional constant D depends on the Reynolds number of the flow. To distinguish between these two possible behaviours, there is need of data with heavy particle pairs having similarly small initial distance, but recorded over a longer time integration and possibly at higher Reynolds numbers flow.

5. Probability density function of inertial particle separation

We now discuss the shape of the PDF for both light and heavy inertial particles. We focus on the time and scale behaviour of the non-stationary PDF

$$\mathscr{P}_{St,\beta}(R, t|R_0, t_0), \tag{5.1}$$

defined as the probability to find a pair of inertial particles (St, β), with separation R at time t, given their initial separation R_0 at time t_0. The case of tracers ($St = 0$, $\beta = 1$) has been widely studied in the past, either experimentally, numerically or theoretically for two- and three-dimensional turbulent flows (see Richardson 1926; Batchelor 1952; Jullien, Paret & Tabeling 1999; Boffetta & Sokolov 2002; Biferale *et al.* 2005; Bourgoin *et al.* 2006; Salazar & Collins 2009). Following the celebrated ideas of Richardson, phenomenological modelling in terms of a diffusion equation for the PDF of pair separation leads to the well-known non-Gaussian distribution

$$\mathscr{P}_{St=0,\beta=1}(R,t) \propto \frac{R^2}{\left(\varepsilon^{1/3}t\right)^{9/2}} \exp\left[-\frac{A\,R^{2/3}}{\varepsilon^{1/3}\,t}\right], \tag{5.2}$$

which is valid for times within the inertial range $\tau_\eta \ll t \ll T_L$ and is obtained assuming a small enough initial separation and statistical homogeneity and isotropy of the three-dimensional turbulent flow. Here, A is a normalization constant. This prediction is based on the simple assumption that, for inertial range distances, tracers undergo a diffusion dynamics with an *effective*, self-similar, turbulent diffusivity given by $K(R,t) \propto R\,\delta_R u \sim \varepsilon^{1/3} R^{4/3}$. Moreover, it relies on the phenomenological assumption that tracers separate in a short-time correlated velocity field. Indeed, it is only if the latter is true, that the diffusion equation for the pair separation becomes exact (see Falkovich, Gawędzki & Vergassola 2001). It is worth recalling that Batchelor proposed a different shape for the eddy diffusivity kernel $K'(t) \propto t^2$, which leads to a Gaussian distribution for the separation PDF (Batchelor 1952). This is, however, observed neither in the case of tracers (see e.g. Biferale *et al.* 2005) nor in the case of heavy particles (see figure 11).

As mentioned earlier, the scenario just drawn for tracers may be strongly contaminated by particle inertia. The main modifications are expected to be due to the presence of small-scale caustics for small-to-large Stokes numbers and to preferential concentration. Caustics make the small-scale velocity field not differentiable and not self-similar, as if inertial particles were separating in a rough velocity field whose exponent depends on distance. Preferential concentration leads to inhomogeneous spatial distribution of particles and manifests itself as a sort of *effective compressibility* in the particle velocity field.

There exists a series of stochastic *toy models* for Lagrangian motion of particles in incompressible/compressible velocity fields, where the statistics of pair separation can be addressed analytically. Among these, the so-called Kraichnan ensemble models, where tracer particles move in a compressible, short-time correlated, homogeneous and isotropic velocity field, with Gaussian spatial correlations (see the review by Falkovich *et al.* 2001 for a description of this model). It is useful in the following to recall two main results obtained for relative dispersion in a Kraichnan compressible flow. The compressibility degree $\wp$ of a velocity field $\boldsymbol{u}$ is defined as the ratio $\wp \equiv \mathscr{C}^2/\mathscr{S}^2$, where $\mathscr{C}^2 \propto \langle(\nabla\cdot\boldsymbol{u})^2\rangle$ and $\mathscr{S}^2 \propto \langle(\nabla\boldsymbol{u})^2\rangle$, and varies between $\wp = 0$ for incompressible flows and $\wp = 1$ for potential flows. We denote by $\wp$ the compressibility degree and by $0 \leqslant \xi \leqslant 2$ the scaling exponent of the two-point velocity correlation function $D_{ij}(r)$ at the scale r, in d-dimensions: $D_{ij}(r) = \langle[u_i(\boldsymbol{r}) - u_i(0)][u_j(\boldsymbol{r}) - u_j(0)]\rangle \sim G_1 r^\xi[(d-1+\xi-\wp\xi)\delta_{ij} + \xi(\wp\,d-1)r_i r_j/r^2]$. For tracer particles moving in such rough flows ($0 \leqslant \xi < 2$), it is possible to show that the pair separation PDF follows a Richardson-like behaviour:

$$\mathscr{P}_{\xi,\mu}(R,t) \propto \frac{R^{D_2-1}}{t^{(d-\mu)/(2-\xi)}} \exp\left[-A\frac{R^{2-\xi}}{t}\right], \quad 0 \leqslant \xi < 2. \tag{5.3}$$

Here, the exponent in the time dependence is $\mu = \wp\xi(d+\xi)/(1+\wp\xi)$, and $D_2 = d - \mu$ is the correlation-dimension, characterizing the fractal spatial distribution of particles. A different distribution emerges when the d-dimensional Kraichan flow is differentiable, i.e. for $\xi = 2$; in such a case, a log-normal PDF is expected for the tracers dispersion:

$$\mathscr{P}_{\xi,\mu}(R, t|R_0, t_0) \propto \frac{1}{R} \exp\left[-\frac{(\log(R/R_0) - \lambda(t-t_0))^2}{2\Delta(t-t_0)}\right], \tag{5.4}$$

with $\Delta = 2G_1(d-1)(1+2\wp)$ and $\lambda = G_1(d-1)(d-4\wp)$. It is worth noting that in the latter case, since the flow is differentiable, the large-time PDF depends on the initial data.

The problem of inertial particle separation in a real turbulent flow presents not only some similarities with the previous toy cases but also important differences.

First, the effective degree of compressibility—due to preferential concentration of inertial particles—is properly defined only in the dissipative range of scales of the fluid flow. For $r \ll \eta$, it is equal to the correlation dimension D_2 defined as $p(r) \sim r^{D_2}$, where $p(r)$ is the probability to find two particles at distance smaller than r, with $r \ll \eta$. As numerically shown by Bec *et al.* (2007*a*) and Calzavarini *et al.* (2008*b*) for three-dimensional turbulent flows, the correlation dimension D_2 depends only on the degree of inertia (St, β), while it does not seem to depend on the Reynolds number of the flow. For $r \gg \eta$, the effective degree of compressibility is, however, no longer constant, but varies with the scale.

Second, in a real turbulent flow, the advecting velocity field exhibits spatial and temporal correlations that are much more complex than those in a Gaussian short-correlated field. Such correlations lead to non-trivial overlaps between particle dynamics and the carrying flow topology. As a result, it is not possible to simply translate the analytical findings obtained in the compressible Kraichnan ensemble to the case of inertial particles; we may expect, however, that in some limits the compressible Kraichnan results should also give the leading behaviour for the case of inertial particles in real turbulent flows studied here.

Within this framework, we first note that the separation PDF valid in the rough case, (5.3), has an asymptotic stretched-exponential decay that is independent on the compressibility degree. This suggests that inertial particle PDF $\mathscr{P}_{St,\beta}(R, t|R_0, t_0)$ must recover the Richardson tracer behaviour of (5.2) in the limit of large scales and large times. Coherent with the discussion in previous sections, for large times and for scales larger than $R^*(St)$, we expect that the heavy pairs PDF (in the limit $\beta \sim 0$) recovers a tracer like distribution:

$$\mathscr{P}_{St,0}(R, t) \sim \exp\left[-A\frac{R^{2/3}}{\varepsilon^{1/3}t}\right], \quad R \gg R^*(St). \tag{5.5}$$

For pairs of light particles, there is no straightforward formulation of such a prediction; as we shall see in the following, preferential concentration effects have a strong fingerprint on the separation PDF even at large times and large scales.

In the opposite limit of very small separations, i.e. $R \ll \eta$, one can correctly assume that the effective degree of compressibility is constant. Therefore, either the small-scale limit for rough flows (5.3) or the small-scale limit for smooth flows (5.4) should be applied, depending on the scaling properties of the particle velocity increment entailed in the value of the exponent $\gamma(St)$, defined from (3.4) and related to the caustics. In

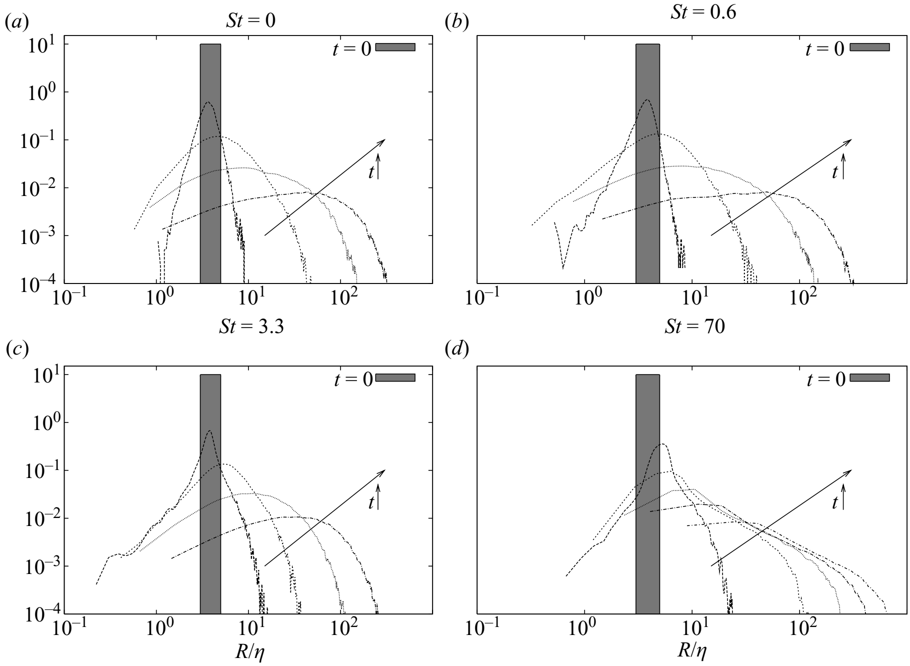

FIGURE 10. Separation PDF, $\mathscr{P}_{St}(R, t|R_0, t_0)$, for heavy pairs with different Stokes numbers at changing time. Initial distance is taken $R_0 \in [3:4]\,\eta$ for $St = 0$ (*a*), $St = 0.6$ (*b*) and $St = 3.3$ (*c*) of Run I, and equal to $R_0 \in [4:6]\,\eta$ for $St = 70$ (*d*) of Run II. The related initial distributions are pictorially depicted with a grey area. Times shown are: $(t - t_0)/\tau_\eta = 1, 6, 18, 36$ for Run I and $(t - t_0)/\tau_\eta = 1, 6, 18, 36, 86$ for Run II.

the former case, it is to be expected to have

$$\mathscr{P}_{St,\beta}(R, t|R_0, t_0) \sim R^{D_2-1}G(t), \quad \text{if } \gamma(St) \neq 1, \tag{5.6}$$

while in the latter,

$$\mathscr{P}_{St,\beta}(R, t|R_0, t_0) \sim R^{D_2/2-1}F(t), \quad \text{if } \gamma(St) = 1. \tag{5.7}$$

Here, F and G are two different decaying functions of time t, whose expressions can be easily derived from (5.3)–(5.4). Note that for the smooth case, i.e. the small-scale limit of the log-normal distribution (5.4), we get for the spatial dependency a factor $D_2/2$ instead of the factor D_2 obtained in the rough case. This will matter in the case of light particle separation, where, due to strong preferential concentration, the probability of finding pairs at a very small distances is large enough to allow for a detailed test of the predictions (5.6) and (5.7). The case of light particles are discussed in §6, while we now turn to a discussion of the above scenario in the case of heavy particle pairs.

5.1. *Probability density function of heavy particle relative separation*

We start by analysing the qualitative evolution of $\mathscr{P}_{St}(R, t)$ at changing time, for different Stokes numbers and in the limit $\beta = 0$ (which is thus omitted from the notation). The four panels of figure 10 show the evolution of the PDF at different times for pairs with $St = 0$ (tracers), and for heavy particles with $St = 1$, 3.3 and 70.

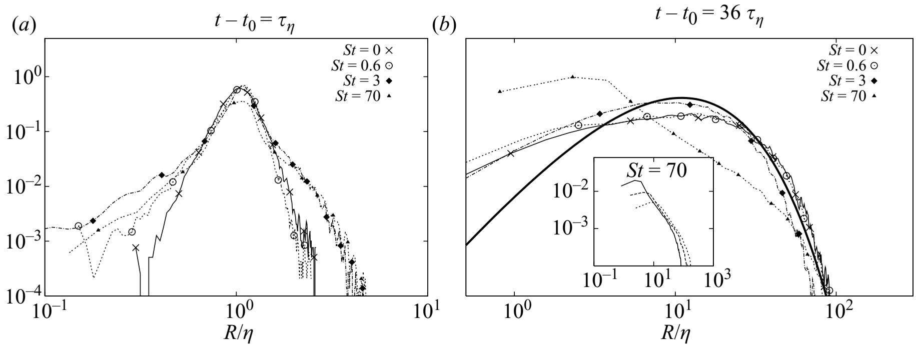

FIGURE 11. Comparison of PDFs at fixed times with data of figure (10). (*a*) Early stage of the separation process, $t - t_0 = \tau_\eta$. Inertia does not affect small Stokes, $St = 0.6$, while its effect is detectable for $St = 3$ and $St = 70$. (*b*) PDF comparison at a later time, $t - t_0 = 36\tau_\eta$. Now, the PDF shows some deviations from the tracer behaviour only for $St = 70$. In (*b*) the solid line gives the Richardson shape (5.2). Initial separation and Reynolds numbers are the same as for figure 10. Inset, PDF evolution for $St = 70$ at three times, $(t - t_0)/\tau_\eta = 36, 82, 130$.

Initially, at $t = t_0$, all selected pairs are separated by the same distance ($R_0 \in [3:6]\,\eta$); this initial distribution is represented in each figure by a grey area. As time elapses, particles separate and their distance reaches different scales, depending on their inertia. Qualitatively, the PDF evolution is very similar for all moderate Stokes numbers: indeed, the PDFs at different moderate Stokes numbers become more and more similar with time. However, in the case of $St = 70$—for which the associate Stokes time τ_s falls well inside the inertial range—the PDF shows a long exponential tail for intermediate separation, which tends to persist at all observed times. To better appreciate such differences, in figure 11 we show the comparison between the different PDFs corresponding to various Stokes numbers for two different times: at the beginning of the separation process, $(t - t_0) = \tau_\eta$, and at a later time, $(t - t_0) = 36\tau_\eta$. As one can see, it is only at early times that the PDFs for moderate-to-large Stokes, $St = 3, 70$ differ in a sensible way from the tracers. In particular, one can clearly see that many pairs have separations much larger and much smaller than those observed for tracers or for heavy pairs with small Stokes numbers. The right tails, describing pairs that are very far apart, are just the signature of the *scrambling* effect of caustics. Such strong events are not captured by second-order moments of the separation statistics that we discussed before, while they clearly affect higher-order moments. The left tails, associated with pairs much closer than tracers, are possibly because of particles that separate at a slower rate than tracers because of preferential concentration induced by inertia.

Later in the evolution, for $(t - t_0) = 36\tau_\eta$, only the separation PDF for $St = 70$ still shows important departure from the tracer case; for all the other Stokes numbers, pairs have had enough time to forget their initial distribution and have practically relaxed on the typical Richardson-like distribution, ruling out the possibility to have a Batchelor-like Gaussian distribution for large pair separations. In the inset, we also show the persistence in the exponential behaviour for the PDF at $St = 70$, by superposing the shapes measured at three times during the particles separation.

With the present data, the small-scale asymptotic behaviour (5.7) cannot be validated for heavy particles. This is because of the limited statistics: very soon after the initial time t_0, there are almost no pairs left with separations $R \ll \eta$.

5.2. *Probability density function of heavy particle relative velocities*

At moderate to large Stokes numbers, the separation process of heavy particle pairs is largely influenced by the presence of large velocity differences at small scales, that is by the presence of caustics in the particles velocity field. In § 3, we have studied stationary statistics (only first-order moment) of velocity differences between heavy particles at changing the distance between particles and their inertia. However, it is also informative to look at the non-stationary, time-dependent distribution of velocity differences and, more particularly, to its distribution measured along heavy pairs separation. The relative velocity $\delta_R \boldsymbol{V}(t) = \dot{\boldsymbol{X}}_1(t) - \dot{\boldsymbol{X}}_2(t)$ can be decomposed into the projection along the separation vector, and two transversal components, here equivalent since the system is statistically isotropic. For tracer particles, the statistics of relative velocity and the alignment properties of $\delta_R \boldsymbol{V}(t)$ and $\boldsymbol{R}(t)$ have been discussed extensively (see e.g. Yeung & Borgas 2004). Here, we focus on the PDF of the relative longitudinal velocity only, which we denote by $\mathscr{W}_{St}(v, t)$, where $v(t) = [\dot{\boldsymbol{X}}_1(t) - \dot{\boldsymbol{X}}_2(t)] \cdot \hat{\boldsymbol{R}}(t)$. For pairs of tracers ($St = 0$), the initial longitudinal velocity distribution is nothing else than the PDF of Eulerian longitudinal velocity increments measured at the distance R_0. For pairs of inertial particles, this initial PDF clearly coincides with the stationary distribution of velocity differences between particles that are at a distance $R = |\boldsymbol{X}_1(t_0) - \boldsymbol{X}_2(t_0)| \in [R_0 : R_0 + \mathrm{d}R_0]$. Such a distribution has the signature of two mechanisms: (i) at small Stokes numbers, only preferential concentration matters and particles probe only a subset of all possible fluid velocity fluctuations; (ii) at large Stokes numbers, particles are homogeneously distributed in the fluid, but they have velocity statistics which may be strongly different from the underlying fluid velocity.

For what concerns heavy pairs, the first effect has not an important signature on small-scale quantities. However, the second effect clearly becomes visible for moderate-to-high inertia as shown in figure 12. Here, we report the longitudinal velocity distributions for pairs with initial distance $R_0 \in [4:6]\,\eta$ and with $St = 0, 1, 3.3$ and 70; the Reynolds number of the underlying flow is $Re_\lambda \sim 400$. Each panel contains the PDFs measured at different times spanning all turbulent time scales. At $t = t_0$, the importance of caustics is manifest for the two largest Stokes numbers, leading to fat tails towards both small and large velocity differences. Interestingly enough, the left tail of $\mathscr{W}_{St}(v, t)$, which describes approaching events of particle relative motion, is immediately dumped already at $(t - t_0) \sim \tau_\eta$; at the same time, however, the right tail continues to be quite fat for the two largest Stokes numbers under consideration. At later stages of the separation process, the tendency of large Stokes pairs to wash out approaching events becomes even stronger. Indeed, at time $(t - t_0) = 38\,\tau_\eta$, the small velocity increments (left) tail has almost disappeared for pairs with $St = 70$. It is worth noting that at those times (i.e. also at those typical scales), heavy particle velocity differences have already started to be smaller than the tracer velocity increments: the larger is the Stokes number, the less pronounced are the PDF tails.

Summarizing, because of the different effects of inertia, we observe a very complex evolution for the longitudinal relative velocity fluctuations along the trajectories of heavy particle pairs. This is certainly a key issue to be considered for stochastic modelling; here, as in a standard *kinetic* problem, both particle positions and velocities need to be modelled to quantitatively control the relative dispersion process.

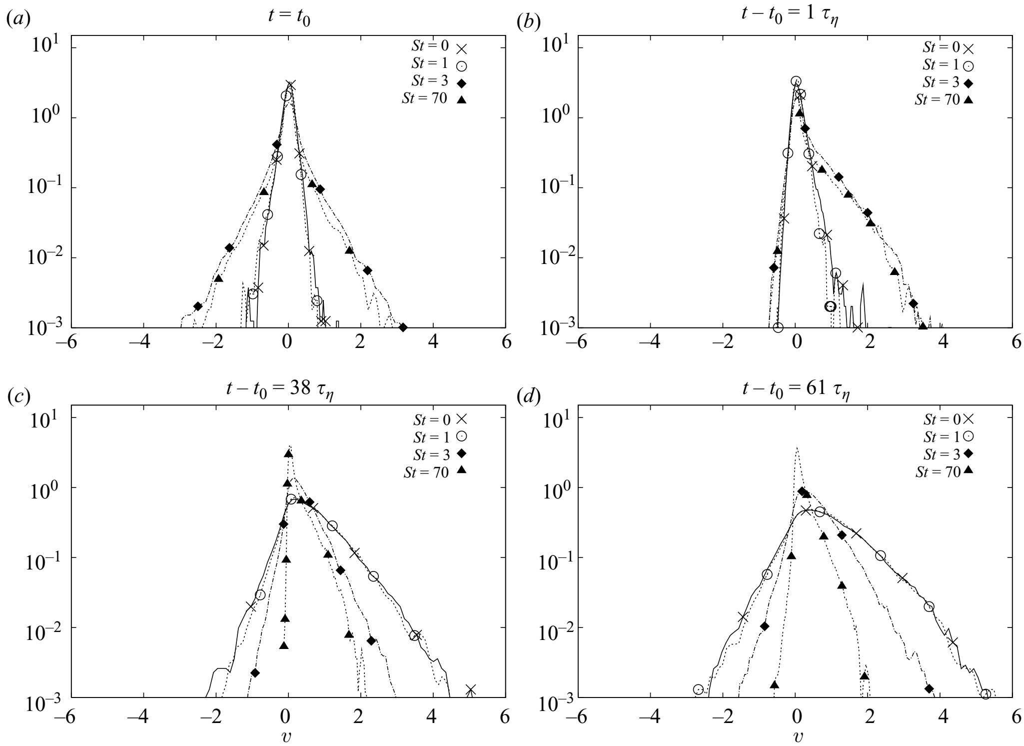

FIGURE 12. Time evolution of the probability density function of heavy particle longitudinal relative velocity, $\mathscr{W}_{St}(v, t)$, during the separation process. Data refer to four different cases: tracers pairs $St = 0$ and heavy pairs $St = 1, 3, 70$, starting with initial distance $R_0 \in [4 : 6]\eta$. PDFs are measured at times $(t - t_0) = [0, 1, 38, 61]\tau_\eta$, for Run II. Note the presence of intense velocity fluctuations for moderate-to-strong inertia, $St = 3, 70$, observable at the early stage of the separation process. These are the legacy of the caustics distribution.

6. Relative dispersion for light particles

So far we have considered the relative motion of very heavy particle pairs, for which the density contrast β with the underlying fluid is zero. In this section, we present results on light particles dynamics as described by (2.2), for different possible choices of the parameters (St, β).

We discuss how the strong effect of preferential concentration—typically observed in the case of light particles in turbulent flows—might influence the intermediate and long-time behaviour of pairs separation. In three-dimensional turbulent flows, as we consider here, light particles associated with different values of (St, β) have been observed to always possess a positive largest Lyapunov exponent (Calzavarini *et al.* 2008*b*); this implies that light pairs always separate in three-dimensional real turbulent flows. We recall, however, that this is not generally true; for instance, in smooth two-dimensional random flows, there are values of (St, β) for which the largest Lyapunov exponent can become negative and particles form pointwise clusters (see Bec 2003).

Light particles with moderate inertia and high density ratio (order-unity St and $\beta = 3$) initially tend to separate much slower than heavy particles with the similar Stokes number; this is evident from the much smaller values of the Lyapunov exponents measured for light particles, with respect to those measured for heavy

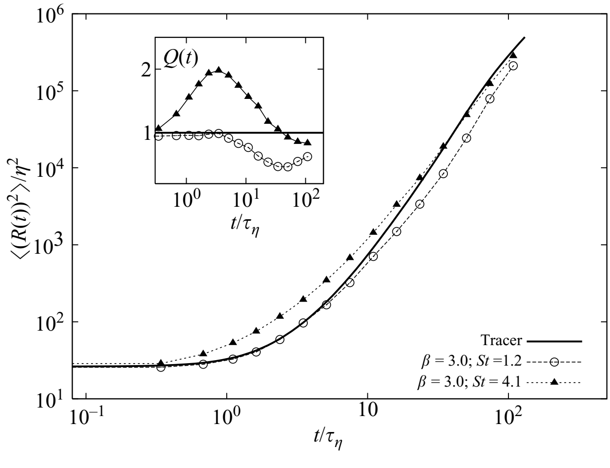

FIGURE 13. Time evolution of the mean-square separation for two different families of light particles ($St = 1.2$, $\beta = 3$) and ($St = 4.1$, $\beta = 3$). The case of tracers is also shown for comparison. Note that the strong small-scale clustering does not affect the long-time behaviour, except through a very small asymptotic slow down. (Inset) Ratio between the mean-square separation for light pairs and that of tracers.

particles with equivalent Stokes numbers but $\beta = 0$. Moreover, finite-time Lyapunov exponents show large fluctuations, indicating that there are pairs that tend to not separate even at long times. Results of this issue will be reported elsewhere. Clearly, pairs of light particles that do not separate will not influence the mean-square distances; hence, we do not expect, and indeed do not measure, any large differences for the long-time behaviour of $\langle |\boldsymbol{R}(t)|^2 \rangle_{St,\beta}$ for light particles, with respect to the heavy case (see figure 13).

It is natural to ask if the light particles' strong preferential concentration might affect high-order moments of the relative separation of two initially close particles and, particularly, the left tail of the separation PDF. Figure 14 shows the time evolution of the separation PDF, $\mathscr{P}_{St,\beta}(R, t|R_0, t_0)$. Data refer to a case with very intense preferential concentration effects and minor influence of caustics ($St = 1.2$, $\beta = 3$) and a case with milder inhomogeneities in the spatial distribution ($St = 0.3$, $\beta = 2$). The initial separation PDF was chosen in both cases by selecting particle pairs with initial distance $R_0 \in [4 : 6]\, \eta$. A remarkable observation is a strong tendency to fill small separations. In other words, there are many pairs that reduce their mutual distance even for very long times. The development of the left tail for the strong clustering case ($St = 1.2$, $\beta = 3$), figure 14(b), is consistent with the estimate given by the long-time, small-scale asymptotic expansion of the log-normal distribution (5.7), $\mathscr{P}(R, t) \sim R^{D_2/2-1}$ for smooth flows, as shown by the straight line in the plot. This confirms that the small-scale dynamics of the highly clustered light particles evolves as that of tracers moving in a smooth, compressible flow (characterized by the same D_2). We also remark that if there is high spatial preferential concentration, caustics cannot be important. This may have important consequences for the estimation of collision kernel of light particles. The approaching events, shown by the left tail in figure 14, are clearly due to the preferential concentration inside vortex-like structures, typical of light particles. Figure 14(a) shows a different case, where preferential concentration is less important, leading to a correlation dimension $D_2 = 2$. Of course, also in this

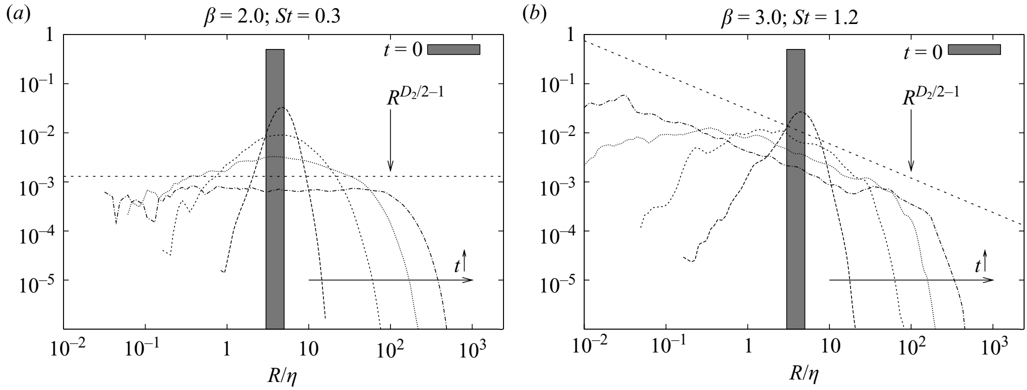

FIGURE 14. Time evolution of the separation PDFs of light particles. (a) $St = 0.3, \beta = 2$, corresponding to a case where preferential concentration is not very effective ($D_2 = 2$). As time elapses, one observes a self-similar filling towards smaller separations, in agreement with (5.7) (dashed line). (b) $St = 1.2$, $\beta = 3$, associated with light particles with strong clustering properties (correlation dimension $D_2 = 0.8$). The self-similar filling of small scales is consistent with the prediction (5.7) with the exponent $D_2/2 - 1$ as depicted by the dashed straight line.

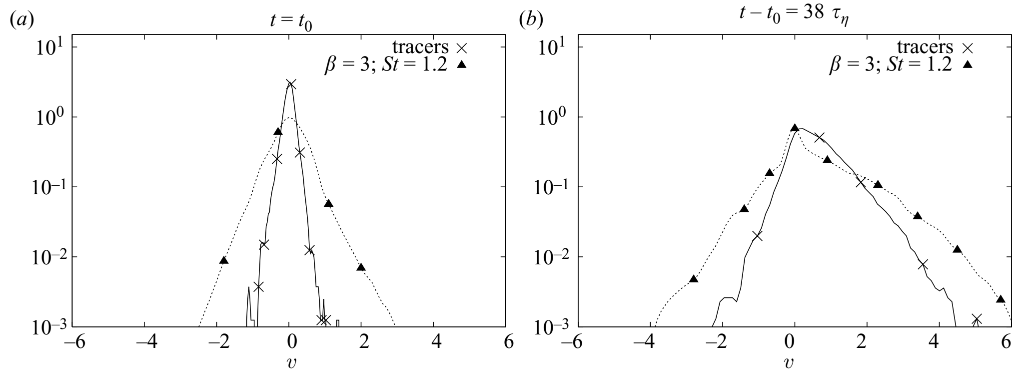

FIGURE 15. PDFs of relative longitudinal velocity, $\mathscr{W}_{St,\beta}(v,t)$, for light pairs with $St = 1.2, \beta = 3$, and for tracers ($St = 0, \beta = 1$). PDFs are measured along the separation process at two different times: (a) initial time, $t = t_0$, (b) $t - t_0 = 38\tau_\eta$.

latter case, there are events with approaching pairs, but these become less and less probable with time.

The importance of preferential concentration can also be appreciated by looking at the PDFs of longitudinal velocity differences between light particles during the separation. We show such distributions for one of the pairs (families) considered above and compare them with those of the tracers (see figure 15). The important difference between the two cases stems from the highly peaked nature of the relative velocity PDF for the strong clustered light particle case. The presence of many pairs with almost vanishing velocity differences, see figure 15(b), is the signature of a coherent bunch of pairs moving in a strongly clustered set.

7. Conclusions

We have studied the relative dispersion of inertial particles in homogeneous and isotropic turbulence from two DNSs at resolutions 512^3 and 2048^3, corresponding to $Re_\lambda \sim 200$ and $Re_\lambda \sim 400$, respectively. We have analysed both heavy and light particle statistics at changing the Stokes numbers. We have studied the evolution of mean separations and the whole PDFs' shape, both for particle distance and velocity increments at changing time and for different typical initial distances. The main results that we have discussed can be summarized as follows. Separations of very heavy particles, with Stokes times falling in the inertial range of the underlying fluid, are strongly affected by the presence of caustics in the initial velocity distribution up to times, when the pair distance reaches scales large enough for the separation dynamics to be again dominated by the underlying flow velocity. As a consequence, a strong transient departure from the Richardson diffusion, with a faster ballistic regime, is observed. A statistical closure of the equation of motions for heavy particle separation is also developed. This model is able to reproduce the main numerical findings.

For light particles, at high density ratio, we observe strong small-scale clustering properties, leading to a considerable fraction of pairs that do not separate at all—although the maximum Lyapunov exponent remains positive. In such a case, the non-stationary spatial concentration at small scales tends to be higher than the analogous case but with a stationary distribution of particles. Such numerical findings open the way to experimental verifications and gives input to the community involved in modelling inertial particle diffusion in applied configurations.

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